

# A Weyl-covariant tensor calculus

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## Abstract

On a (pseudo-) Riemannian manifold of dimension  $n \geq 3$ , the space of tensors which transform covariantly under Weyl rescalings of the metric is built. This construction is related to a Weyl-covariant operator  $\mathcal{D}$  whose commutator  $[\mathcal{D}, \mathcal{D}]$  gives the conformally invariant Weyl tensor plus the Cotton tensor. So-called generalized connections and their transformation laws under diffeomorphisms and Weyl rescalings are also derived. These results are obtained by application of BRST techniques.

## 1 Introduction

Recently [1], a purely algebraic method was used to solve the problem of constructing and classifying all the local scalar invariants of a conformal structure on a (pseudo-) Riemannian manifold of dimension  $n = 8$ . The approach, however, is not confined to  $n = 8$ , and one of the purposes of this paper is to explain the derivation of the so-called Weyl-covariant tensors, the building blocks of the local conformal invariants in arbitrary dimension  $n \geq 3$ .

In the context of local gauge field theory, the determination of quantities which are invariant under a given set of gauge transformations can be rephrased in terms of local BRST cohomology. Within the BRST framework, the gauge symmetry and its algebra are encoded in a single differential  $s$  [2]. Powerful techniques for the computation of BRST cohomologies are proposed in [3] (see also [4]), that apply to a large class of gauge theories and relate the BRST cohomology to an underlying gauge covariant algebra. At the core of this analysis is a definition of tensor fields and connections on which an underlying gauge covariant algebra is realized. Such a characterization of tensor fields, connections and the corresponding transformation laws has the advantage that it is purely algebraic and does not invoke any concept in addition to the BRST cohomology itself.

In the present paper we consider theories where the only classical field is the metric  $g_{\mu\nu}$  and the gauge symmetries are diffeomorphisms plus Weyl rescalings. Explicitly, the

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infinitesimal gauge transformations are

$$\delta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu} + \delta_\phi^W g_{\mu\nu} = \zeta^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \zeta^\rho g_{\rho\nu} + \partial_\nu \zeta^\rho g_{\mu\rho} + 2\phi g_{\mu\nu}. \quad (1.1)$$

Along the lines of [3], we construct the space  $\mathcal{W}$  of tensors and generalized connections that transform covariantly with respect to diffeomorphisms and Weyl transformations. The latter property means that, under Weyl rescalings, the tensors belonging to  $\mathcal{W}$  make appear at most the first derivative  $\partial_\mu \phi$  of the Weyl parameter  $\phi$ , and no derivative  $\partial_{\mu_1} \dots \partial_{\mu_k} \phi$  with  $k \geq 2$ .

Knowing the space  $\mathcal{W}$ , we are able to define an operator  $\mathcal{D}$  acting in  $\mathcal{W}$  and such that  $[\mathcal{D}, \mathcal{D}] \sim C + \tilde{C}$ , where  $C$  and  $\tilde{C}$  respectively denote the conformally invariant Weyl tensor and the Cotton tensor. The Weyl-covariant derivative  $\mathcal{D}$  generates the whole space of tensor fields belonging to  $\mathcal{W}$  by successive applications on  $C$  (and  $\tilde{C}$  in  $n = 3$ ). The rule for the commutator  $[\mathcal{D}, \mathcal{D}]$  is at the basis of the Weyl-covariant tensor calculus utilized in [1]. Other useful relations are obtained which are nothing but the Jacobi identities for the underlying gauge covariant algebra alluded to before.

The generalized connections play no rôle in the construction of local Weyl invariants, but are of prime importance in many other issues, like for example in the determination of the counterterms, the consistent interactions and the conservation laws that a gauge theory admits. They are also relevant for the classification of the Weyl anomalies, the solutions of the Wess-Zumino consistency condition for a theory describing conformal massless matter fields in an external gravitational background. The latter problem amounts to the computation of a BRST cohomology group and will be analyzed elsewhere.

## 2 BRST formulation

### 2.1 Some definitions

As mentioned above, the derivation of the space  $\mathcal{W}$  of Weyl-covariant tensors and generalized connections is purely algebraic and requires no dynamical information.<sup>2</sup> As a consequence, all what we need is contained in equation (1.1) and the BRST differential  $s$  reduces to  $\gamma$ , the differential along the gauge orbits. We refer to [6, 7] for more details on the BRST formalism as used throughout the present work.

A  $\mathbb{Z}$ -grading called *ghost number* is associated to the differential  $\gamma$ . The latter raises the ghost number by one unit and is decomposed according to the degree in the Weyl ghost (the fermionic field associated to the Weyl parameter):  $\gamma = \gamma_0 + \gamma_1$ . The first part  $\gamma_0$  contains the

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<sup>2</sup>For a BRST-cohomological derivation of Weyl gravity in the Batalin-Vilkovisky antifield formalism, see [5].

information about the diffeomorphisms. The second part,  $\gamma_1$ , corresponds to Weyl rescalings of the metric and increases the number of (possibly differentiated) Weyl ghosts by 1.

The action of  $\gamma$  on the fields  $\Phi^A$  (including the ghosts) is given as follows

$$\gamma_0 g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}, \quad \gamma_1 g_{\mu\nu} = 2\omega g_{\mu\nu}, \quad (2.2)$$

$$\gamma_0 \xi^\mu = \xi^\rho \partial_\rho \xi^\mu, \quad \gamma_0 \omega = \xi^\rho \partial_\rho \omega, \quad \gamma_1 \xi^\mu = 0, \quad \gamma_1 \omega = 0. \quad (2.3)$$

The field  $\omega$  is the Weyl ghost, the anticommuting field associated to the Weyl parameter  $\phi$ , while  $\xi^\mu$  is the anticommuting diffeomorphisms ghost associated to the vector field  $\zeta^\mu$  of equation (1.1). By definition, the Grassmann-odd fields  $\omega$  and  $\xi^\mu$  have ghost number  $+1$ . The last equality of (2.3) reflects the abelian nature of the algebra of Weyl transformations. From the above equations and by using the fact that  $\gamma$  is an odd derivation, it is easy to check that  $\gamma$  is indeed a differential.

One unites the BRST differential  $\gamma$  and the total exterior derivative  $d$  into a single differential  $\tilde{\gamma} = \gamma + d$ . Then, the Wess-Zumino consistency condition and its descent are encapsulated in

$$\tilde{\gamma} \tilde{a} = 0, \quad \tilde{a} \neq \tilde{\gamma} \tilde{b} + \text{constant} \quad (2.4)$$

for the local total forms  $\tilde{a}$  and  $\tilde{b}$  of total degrees  $G = n+1$  and  $G = n$  [3]. Total local forms are by definition formal sums of local forms with different form degrees and ghost numbers:  $\tilde{a} = \sum_{p=0}^n a_p^{G-p}$ , where subscripts (resp. superscripts) denote the form degree (resp. the ghost number). A local  $p$ -form  $\omega_p$  depends on the fields  $\Phi^A$  and their derivatives up to some finite (but otherwise unspecified) order, which is denoted by  $\omega_p = \frac{1}{p!} dx^{\mu_1} \dots dx^{\mu_p} \omega_{\mu_1 \dots \mu_p}(x, [\Phi^A])$ .

The equations (2.4) imply that  $\tilde{a}$  is a non-trivial element of the cohomology group  $H(\tilde{\gamma})$  in the algebra of total local forms. As shown in [3], the cohomology of  $\gamma$  in the space of local functionals (integrals of local  $n$ -forms) is indeed locally isomorphic to the cohomology of  $\tilde{\gamma}$  in the space of local total forms. In other words, the solutions  $a_n^g$  of the Wess-Zumino consistency condition

$$\gamma a_n^g + da_{n-1}^{g+1} = 0, \quad a_n^g \neq \gamma b_n^{g-1} + db_{n-1}^g \quad (2.5)$$

correspond one-to-one (modulo trivial solutions) to the solutions  $\tilde{a}$  of (2.4) at total degree  $G = g + n$ ,  $\text{totdeg}(\tilde{a}) = g + n$ .

The solutions of (2.4) or (2.5) determine the general structure of the counterterms that an action admits, the possible gauge anomalies, the conserved currents, the consistent interactions, *etc.* [7]. In the next sections and in the appendix, we determine the restricted space  $\mathcal{W}$  of the space of total local forms in which these solutions naturally appear, for a theory invariant under the transformations (1.1).

We close this section with some definitions and conventions. The conformally invariant Weyl tensor  $C^\beta_{\gamma\delta\varepsilon}$  and the tensor  $K_{\alpha\beta}$  are given by

$$C^\alpha_{\beta\gamma\delta} := R^\alpha_{\beta\gamma\delta} - 2(\delta^\alpha_{[\gamma} K_{\delta]\beta} - g_{\beta[\gamma} K_{\delta]}^\alpha), \quad (2.6)$$

$$K_{\alpha\beta} := \frac{1}{n-2} \left( R_{\alpha\beta} - \frac{1}{2(n-1)} g_{\alpha\beta} R \right). \quad (2.7)$$

The Ricci tensor is  $R_{\beta\delta} = R^\alpha_{\beta\alpha\delta}$ , where  $R^\alpha_{\beta\gamma\delta} = (\partial_\gamma \Gamma_{\beta\delta}^\alpha + \Gamma_{\gamma\lambda}^\alpha \Gamma_{\beta\delta}^\lambda) - (\gamma \leftrightarrow \delta)$  is the Riemann tensor. The Christoffel symbols are given by  $\Gamma_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\lambda} (\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta})$ . Curved brackets denote strength-one complete symmetrization, whereas square brackets denote strength-one complete antisymmetrization. We have  $\nabla_\mu g_{\alpha\beta} = 0$ , where the symbol  $\nabla$  denotes the usual torsion-free covariant derivative associated to  $\Gamma_{\alpha\beta}^\gamma$ . Finally, the derivative  $\partial_\alpha \omega$  of the Weyl ghost will sometimes be noted  $\omega_\alpha \equiv \partial_\alpha \omega$ .

## 2.2 Contracting homotopy

A well-known technique in the study of cohomologies is the use of contracting homotopies. The idea is to construct contracting homotopy operators which allow to eliminate certain local jet coordinates, called *trivial pairs*, from the cohomological analysis. This reduces the cohomological problem to an analogous one involving only the remaining jet coordinates. For that purpose one needs to construct suitable sets of jets coordinates replacing the fields, the ghosts and all their derivatives and satisfying appropriate requirements.

The lemma at the basis of the contracting homotopy techniques is, in the notations of [3] to which we refer for more details,

**Lemma 1.** *Suppose there is a set of local jet coordinates*

$$\mathcal{B} = \{\mathcal{U}^\ell, \mathcal{V}^\ell, \mathcal{W}^\Lambda\}$$

*such that the change of coordinates from  $\mathcal{J} = \{[\Phi^A], x^\mu, dx^\mu\}$  to  $\mathcal{B}$  is local and locally invertible and*

$$\begin{aligned} \tilde{\gamma} \mathcal{U}^\ell &= \mathcal{V}^\ell \quad \forall \ell, \\ \tilde{\gamma} \mathcal{W}^\Lambda &= \mathcal{R}^\Lambda(\mathcal{W}) \quad \forall \Lambda. \end{aligned}$$

*Then, locally the  $\mathcal{U}$ 's and  $\mathcal{V}$ 's can be eliminated from the  $\tilde{\gamma}$ -cohomology, i.e. the latter reduces locally to the  $\tilde{\gamma}$ -cohomology on total local forms depending only on the  $\mathcal{W}$ 's.*

Thus, in order to compute and classify the local Weyl-invariant scalar densities [1] or for the solutions of the Wess-Zumino consistency conditions (2.5), it is sufficient to work in the space  $\mathcal{W}$ . In the context of Weyl gravity theories, we have the following

**Proposition 1.** *Let  $\mathcal{J}$  be the jet space  $\mathcal{J} = \{[g_{\mu\nu}], [\omega], [\xi^\mu], x^\mu, dx^\mu\}$  and  $\tilde{\gamma} = \gamma_0 + \gamma_1 + d$  the differential acting on  $\mathcal{J}$  according to*

$$\gamma_0 g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}, \quad \gamma_1 g_{\mu\nu} = 2\omega g_{\mu\nu}, \quad (2.8)$$

$$\gamma_0 \xi^\mu = \xi^\rho \partial_\rho \xi^\mu, \quad \gamma_0 \omega = \xi^\rho \partial_\rho \omega, \quad \gamma_1 \xi^\mu = 0, \quad \gamma_1 \omega = 0. \quad (2.9)$$

*Then, the  $\{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$ -decomposition of  $\mathcal{J}$  corresponding to  $\tilde{\gamma}$  is*

$$\begin{aligned} \{\mathcal{U}^\ell\} &= \{x^\mu, \partial_{(\mu_1 \dots \mu_k} \Gamma_{\mu_{k+1} \mu_{k+2}})^{\nu}, \nabla_{(\mu_1 \dots \mu_k} K_{\mu_{k+1} \mu_{k+2}}), k \in \mathbb{N}\}, \\ \{\mathcal{V}^\ell\} &= \{\tilde{\gamma} \mathcal{U}^\ell\}, \quad \{\mathcal{W}^\Lambda\} = \{\mathcal{T}^i, \tilde{C}^N\}, \\ \{\mathcal{T}^i\} &= \{g_{\mu\nu}, \mathcal{D}_{(\mu_1} \dots \mathcal{D}_{\mu_k} C_{\gamma\delta\varepsilon}^{\beta}, k \in \mathbb{N}\}, \end{aligned} \quad (2.10)$$

$$\begin{aligned} \{\tilde{C}^N\} &= \{2\omega, \tilde{\xi}^\nu, \tilde{C}_\nu{}^\rho, \tilde{\omega}_\alpha\}, \\ \tilde{\xi}^\nu &:= \xi^\nu + dx^\nu, \quad \tilde{C}_\nu{}^\rho := \partial_\nu \xi^\rho + \tilde{\xi}^\alpha \Gamma_{\alpha\nu}{}^\rho, \quad \tilde{\omega}_\alpha := \omega_\alpha - \tilde{\xi}^\beta K_{\alpha\beta}. \end{aligned} \quad (2.11)$$

The rest of the paper contains the definition of the operator  $\mathcal{D}$  together with the  $\tilde{\gamma}$ -transformation rules for the elements of  $\mathcal{W}$ . A remark will also be made for the case  $n = 3$ . The proposition follows then by the fact that *every function of the Riemann tensor and its covariant derivatives can be written as a function of the Weyl tensor and its covariant derivatives plus the completely symmetric tensors  $\nabla_{(\lambda_1 \lambda_2 \dots \lambda_k} K_{\alpha\beta)}$* . A proof of the latter statement can be found in the Appendix A of [1].

It is understood that only the algebraically independent components of  $g_{\mu\nu}$  and  $C_{\gamma\delta\varepsilon}^{\beta}$  enter into (2.10). [Together with the symmetrization of the indices in (2.10), this guarantees the absence of algebraic identities between the generators  $\mathcal{T}^i$ , taking into account the second equation of (2.20) and the Bianchi identity (2.22).]

The tensor fields  $\{\mathcal{T}^i\}$  have total degree zero whereas the generalized connections  $\{\tilde{C}^N\}$  have total degree 1. They decompose into two parts, the first of ghost number 1 and form degree zero, the second of ghost number zero and form degree 1:

$$\begin{aligned} \text{totdeg}(\mathcal{T}^i) &= 0, \quad \text{totdeg}(\tilde{C}^N) = 1, \quad \tilde{C}^N = \hat{C}^N + \mathcal{A}^N, \\ gh(\hat{C}^N) &= 1 = \text{formdeg}(\mathcal{A}^N), \quad gh(\mathcal{A}^N) = 0 = \text{formdeg}(\hat{C}^N), \end{aligned}$$

where, from (2.11),

$$\{\hat{C}^N\} = \{2\omega, \xi^\nu, \hat{C}_\nu{}^\rho := \partial_\nu \xi^\rho + \xi^\alpha \Gamma_{\alpha\nu}{}^\rho, \hat{\omega}_\alpha := \omega_\alpha - \xi^\mu K_{\mu\alpha}\}, \quad (2.12)$$

$$\{\mathcal{A}^N\} = \{0, dx^\mu \delta_\mu^\nu, dx^\mu \Gamma_{\mu\nu}{}^\rho, -dx^\mu K_{\mu\alpha}\}. \quad (2.13)$$

The  $\mathcal{A}^N$ 's and  $\hat{C}^N$ 's are called respectively connection 1-forms and covariant ghosts [3].

Since  $\tilde{\gamma}$  raises the total degree by one unit, we have

$$\tilde{\gamma} \mathcal{T}^i = \tilde{C}^N \Delta_N \mathcal{T}^i \Leftrightarrow \begin{cases} \gamma \mathcal{T}^i = \hat{C}^N \Delta_N \mathcal{T}^i \\ d\mathcal{T}^i = \mathcal{A}^N \Delta_N \mathcal{T}^i \end{cases}, \quad (2.14)$$

$$\{\Delta_N\} = \{\Delta, \mathcal{D}_\nu, \Delta_\rho{}^\nu, \Gamma^\alpha\}. \quad (2.15)$$

## 2.3 BRST covariant algebra for Weyl-gravity

The Weyl-covariant derivative  $\mathcal{D}$  is given by

$$\mathcal{D}_\mu := \partial_\mu - \Gamma_{\mu\nu}{}^\rho \Delta_\rho{}^\nu + K_{\mu\alpha} \Gamma^\alpha. \quad (2.16)$$

The aim of this section is to make precise the above definition by explicitly defining the three operators  $\{\Delta, \Delta_\rho{}^\nu, \Gamma^\alpha\}$  introduced in (2.14) and (2.15). An underlying gauge covariant algebra will be exhibited, which provides a compact formulation of the BRST algebra on tensor fields and generalized connections.

1. The operator  $\Delta$  corresponds to the dimension operator. It counts the number of metrics that explicitly appear in a given expression,

$$\Delta := g_{\mu\nu} \frac{\partial^{expl}}{\partial g_{\mu\nu}}.$$

For example,  $\Delta(g^{\gamma\mu_2} g^{\lambda\mu_1} \mathcal{D}_{\mu_1} C_{\gamma\delta\varepsilon}^\beta) = -2(g^{\gamma\mu_2} g^{\lambda\mu_1} \mathcal{D}_{\mu_1} C_{\gamma\delta\varepsilon}^\beta)$  and  $\Delta(g_{\alpha\beta} g^{\gamma\delta}) = 0$ . As a consequence of (2.14), (2.15) and (2.12), we can write  $\gamma_1 \sqrt{|g|} = 2\omega \Delta \sqrt{|g|} = 2\omega(\frac{n}{2} \sqrt{|g|}) = n\omega \sqrt{|g|}$ , where  $|g|$  denotes the absolute value of the determinant of  $g_{\mu\nu}$  (supposed invertible).

2. The operator  $\Delta_\mu{}^\rho$  generates  $GL(n)$ -transformations of world indices according to

$$\Delta_\mu{}^\nu T_\alpha^\beta = \delta_\alpha^\nu T_\mu^\beta - \delta_\mu^\beta T_\alpha^\nu,$$

where  $T_\alpha^\beta$  is a (1,1)-type tensor under  $GL(n)$  transformations. The usual torsion-free covariant derivative can thus be written  $\nabla_\mu = \partial_\mu - \Gamma_{\mu\nu}{}^\rho \Delta_\rho{}^\nu$ . Note that this expression must be completed by  $p \Gamma_{\mu\alpha}{}^\alpha$  if one takes the covariant derivative  $\nabla_\mu$  of a weight- $p$  tensor density, so  $\nabla = dx^\mu \nabla_\mu = dx^\mu \partial_\mu - \tilde{C}_\nu{}^\rho \Delta_\rho{}^\nu + p \tilde{C}_\mu{}^\mu$ .

3. In order to conveniently define the action of the generator  $\Gamma^\alpha$ , we first define the so-called  $W$ -tensors carrying super-indices  $\Omega_k$ :

$$\begin{aligned} W_{\Omega_0} &:= C_{\gamma\delta\varepsilon}^\beta, \quad W_{\Omega_1} := \mathcal{D}_{\alpha_1} C_{\gamma\delta\varepsilon}^\beta, \quad \dots, \\ W_{\Omega_k} &:= \mathcal{D}_{\alpha_k} \mathcal{D}_{\alpha_{k-1}} \dots \mathcal{D}_{\alpha_2} \mathcal{D}_{\alpha_1} C_{\gamma\delta\varepsilon}^\beta. \end{aligned}$$

Then, we can write  $\{\mathcal{T}^i\} \subset \{g_{\mu\nu}, \{W_{\Omega_k}\} : k = 0, 1, \dots\}$  and the operator  $\Gamma^\alpha$  acts on space of the  $W$ -tensors according to

$$\Gamma^\alpha W_{\Omega_j} = [T^\alpha]_{\Omega_j}^{\Omega_{j-1}} W_{\Omega_{j-1}}, \quad \Gamma^\alpha := [T^\alpha]_{\Omega_i}^{\Omega_{i-1}} \Delta_{\Omega_{i-1}}^{\Omega_i}, \quad (2.17)$$

where  $\Delta_{\Omega_j}^{\Omega_k} W_{\Omega_i} = \delta_{\Omega_i}^{\Omega_k} W_{\Omega_j}$  and where the symbol  $\delta_{\Omega_i}^{\Omega_k}$  is such that  $\delta_{\Omega_i}^{\Omega_k} W_{\Omega_k} = W_{\Omega_i}$ . We use Einstein's summation conventions for the  $W$ -tensor super-indices  $\Omega_i$ . The matrices  $[T^\alpha]_{\Omega_j}^{\Omega_{j-1}}$  are obtained by recursion in the appendix, with  $[T^\alpha]_{\Omega_j}^{\Omega_{j-1}} = 0 \quad \forall j \leq 0$ . The action of  $\Gamma^\alpha$  gives zero on everything but the  $W$ -tensors. In particular,  $\Gamma^\alpha g_{\mu\nu} = 0$ .

The  $W$ -tensors transform under  $\tilde{\gamma}$  according to (2.14), (2.15), (2.12) and (2.13). They are the building blocks for the construction of Weyl invariants [1]. Note that the Bach tensor is nothing but the following double trace of  $W_{\Omega_2}$ :

$$B_{\mu\nu} \equiv \nabla^\alpha \tilde{C}_{\mu\nu\alpha} - K^{\lambda\rho} C_{\lambda\mu\nu} = \frac{1}{(3-n)} g^{\alpha\rho} \mathcal{D}_\alpha \mathcal{D}_\beta C_{\mu\nu\rho}^\beta.$$

The action of  $\tilde{\gamma}$  on the generalized connections is

$$\begin{aligned} \cdot \quad & \tilde{\gamma}\omega = \tilde{\xi}^\mu \tilde{\omega}_\mu, \\ \cdot \quad & \tilde{\gamma}\tilde{\xi}^\mu = \tilde{\xi}^\rho \tilde{C}_\rho^\mu, \\ \cdot \quad & \tilde{\gamma}\tilde{C}_\mu^\nu = \tilde{C}_\mu^\alpha \tilde{C}_\alpha^\nu + \frac{1}{2} \tilde{\xi}^\rho \tilde{\xi}^\sigma C_{\mu\rho\sigma}^\nu + \mathcal{P}_{\mu\beta}^{\alpha\nu} \tilde{\omega}_\alpha \tilde{\xi}^\beta, \\ \cdot \quad & \tilde{\gamma}\tilde{\omega}_\alpha = \frac{1}{2} \tilde{\xi}^\rho \tilde{\xi}^\sigma \tilde{C}_{\alpha\rho\sigma} + \tilde{C}_\alpha^\beta \tilde{\omega}_\beta, \end{aligned}$$

where  $\mathcal{P}_{\mu\beta}^{\alpha\nu} := (-g^{\alpha\nu} g_{\mu\beta} + \delta_\mu^\alpha \delta_\beta^\nu + \delta_\beta^\alpha \delta_\mu^\nu)$  and the tensor  $\tilde{C}_{\alpha\rho\sigma} \equiv \frac{1}{2} \nabla_{[\sigma} K_{\rho]\alpha}$  is the Cotton tensor. Note that  $C_{\nu\alpha\beta}^\mu = R_{\nu\alpha\beta}^\mu - 2\mathcal{P}_{\nu[\alpha}^{\mu\rho} K_{\beta]\rho}$  and  $\gamma_1 \Gamma_{\mu\beta}^\nu = \mathcal{P}_{\mu\beta}^{\alpha\nu} \omega_\alpha$ .

From  $\tilde{\gamma}^2 \mathcal{T}^i = 0$ , we derive the gauge covariant algebra generated by  $\{\Delta, \mathcal{D}_\nu, \Delta_\rho^\nu, \Gamma^\alpha\}$ :

$$[\Delta_\nu^\rho, \Gamma^\alpha] = -\delta_\nu^\alpha \Gamma^\rho, \quad [\Gamma^\alpha, \Gamma^\beta] = 0, \quad (2.18)$$

$$[\Delta_\nu^\rho, \mathcal{D}_\mu] = \delta_\mu^\rho \mathcal{D}_\nu, \quad [\Delta_\mu^\rho, \Delta_\nu^\sigma] = \delta_\nu^\rho \Delta_\mu^\sigma - \delta_\mu^\rho \Delta_\nu^\sigma, \quad (2.19)$$

$$[\Gamma^\alpha, \mathcal{D}_\beta] = -\mathcal{P}_{\mu\beta}^{\alpha\nu} \Delta_\nu^\mu, \quad [\mathcal{D}_\mu, \mathcal{D}_\nu] = C_{\mu\nu\rho}^\sigma \Delta_\sigma^\rho - \tilde{C}_{\alpha\mu\nu} \Gamma^\alpha, \quad (2.20)$$

where the operator  $\Delta$  commutes with everything. The second equality of (2.18) reflects the abelian nature of the Weyl transformations, while the second equality of (2.20) displays the commutator of two Weyl-covariant derivatives in terms of the Weyl tensor and the Cotton tensor. Note that the commutator of two covariant derivatives reads  $[\nabla_\mu, \nabla_\nu] = R_{\mu\nu\rho}^\sigma \Delta_\sigma^\rho$ .

From  $\tilde{\gamma}^2 \tilde{C}^N = 0$ , we find the following set of Bianchi identities

$$\cdot \quad \tilde{\gamma}^2 \omega = 0 \Rightarrow \tilde{C}_{[\mu\rho\sigma]} = 0 \quad (2.21)$$

$$\cdot \quad \tilde{\gamma}^2 \tilde{C}_\mu^\nu = 0 \Rightarrow \nabla_{[\gamma} C_{\delta\varepsilon]\alpha\beta} - \tilde{C}_{\alpha[\gamma\delta} g_{\varepsilon]\beta} + \tilde{C}_{\beta[\gamma\delta} g_{\varepsilon]\alpha} = 0 \quad (2.22)$$

$$\cdot \quad \tilde{\gamma}^2 \tilde{\xi}^\mu = 0 \Rightarrow \begin{cases} \mathcal{P}_{[\rho\nu]}^{\alpha\mu} = 0 \\ C_{[\nu\rho\sigma]}^\mu = 0 \end{cases} \quad (2.23)$$

$$\cdot \quad \tilde{\gamma}^2 \tilde{\omega}_\alpha = 0 \Rightarrow \begin{cases} \Gamma^\alpha \tilde{C}_{\beta\rho\sigma} + C_{\beta\rho\sigma}^\alpha = 0 \\ \mathcal{D}_{[\beta} \tilde{C}_{\rho\sigma]\alpha} = 0 \end{cases} \quad (2.24)$$

which are nothing but the Jacobi identities for the algebra (2.18)–(2.20).

Note that the case  $n = 3$  proceeds in exactly the same way, provided one sets  $C^\mu_{\nu\rho\sigma}$  to zero and one defines  $W_{\Omega_0}^{(3)} := \tilde{C}_{\alpha\rho\sigma}$ . In other words, the relations (2.18)–(2.20) and (2.21)–(2.24) still hold, setting  $C^\mu_{\nu\rho\sigma} = 0$ . The representation matrices  $\mathbf{\Gamma}^\alpha$  and the Weyl-covariant derivative (2.16) are unchanged as well. More explicitly, we have

$$\begin{aligned} \underline{n \geq 4} & : \quad \gamma_1 \mathcal{D}_{\alpha_1} C^\beta_{\gamma\delta\varepsilon} = \omega_\alpha (-\mathcal{P}^{\alpha\nu}_{\mu\alpha_1} \Delta_\nu^\mu) C^\beta_{\gamma\delta\varepsilon} \rightsquigarrow \gamma_1 W_{\Omega_1} = \omega_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_1} \\ \underline{n = 3} & : \quad \gamma_1 \mathcal{D}_{\alpha_1} \tilde{C}_{\gamma\delta\varepsilon} = \omega_\alpha (-\mathcal{P}^{\alpha\nu}_{\mu\alpha_1} \Delta_\nu^\mu) \tilde{C}_{\gamma\delta\varepsilon} \rightsquigarrow \gamma_1 W_{\Omega_1}^{(3)} = \omega_\alpha \mathbf{\Gamma}_{(3)}^\alpha W_{\Omega_1}^{(3)}, \end{aligned}$$

which shows that the representation matrices  $\mathbf{\Gamma}^\alpha$  and  $\mathbf{\Gamma}_{(3)}^\alpha$  are essentially the same. Indeed, the iterative procedure given in the appendix reproduces itself in exactly the same way when  $n = 3$ , with the convention that  $W_{\Omega_0}^{(3)} \equiv \tilde{C}_{\alpha\rho\sigma}$ .

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## A $W$ -tensors and their transformations

The  $W$ -tensors are computed iteratively, together with their transformation laws under Weyl rescalings of the metric.

**(A)** First, we have  $\gamma_1 W_{\Omega_0} = \omega_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_0} = 0$ . Then, we form  $W_{\Omega_1} = \nabla_{\alpha_1} W_{\Omega_0}$ . Taking the Weyl variation gives

$$\begin{aligned} \gamma_1 W_{\Omega_1} &= \gamma_1 [(\partial_{\alpha_1} - \Gamma_{\alpha_1\mu}{}^\nu \Delta_\nu^\mu) W_{\Omega_0}] = -\omega_\lambda \mathcal{P}^{\lambda\nu}_{\mu\alpha_1} \Delta_\nu^\mu W_{\Omega_0} \\ &= \omega_\lambda [T^\lambda]_{\Omega_1}^{\Omega_0} W_{\Omega_0}, \end{aligned}$$

where the last equality serves as a definition for the tensor  $[T^\lambda]_{\Omega_1}^{\Omega_0}$ , which satisfies  $\gamma_1 [T^\lambda]_{\Omega_1}^{\Omega_0} = 0 = \nabla_\mu [T^\lambda]_{\Omega_1}^{\Omega_0}$ . We also use the notation  $\gamma_1 W_{\Omega_1} = \omega_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_0}$ , cf. equation (2.17).

Continuing, we compute the Weyl variation of  $\nabla_{\alpha_2} W_{\Omega_1}$ :

$$\begin{aligned} \gamma_1 \nabla_{\alpha_2} W_{\Omega_1} &= \nabla_{\alpha_2} (\omega_\lambda [T^\lambda]_{\Omega_1}^{\Omega_0} W_{\Omega_0}) - \omega_\lambda \mathcal{P}^{\lambda\nu}_{\mu\alpha_2} \Delta_\nu^\mu W_{\Omega_1} \\ &= (-\gamma_1 K_{\lambda\alpha_2}) [T^\lambda]_{\Omega_1}^{\Omega_0} W_{\Omega_0} + \omega_\lambda [T^\lambda]_{\Omega_1}^{\Omega_0} \nabla_{\alpha_2} W_{\Omega_0} \\ &\quad - \omega_\lambda \mathcal{P}^{\lambda\nu}_{\mu\alpha_2} \Delta_\nu^\mu W_{\Omega_1}. \end{aligned}$$



Using  $\gamma_1([T^\lambda]_{\Omega_1}^{\Omega_0} W_{\Omega_0}) = 0$ , we obtain

$$\begin{aligned} \gamma_1 \left( \nabla_{\alpha_2} W_{\Omega_1} + K_{\lambda\alpha_2} [T^\lambda]_{\Omega_1}^{\Omega_0} W_{\Omega_0} \right) &= \\ &= \omega_\lambda \left( \delta_{\alpha_2\Omega_0}^{\Omega_1'} [T^\lambda]_{\Omega_1}^{\Omega_0} - \delta_{\Omega_1}^{\Omega_1'} \mathcal{P}_{\mu\alpha_2}^{\lambda\nu} \Delta_\nu^\mu \right) W_{\Omega_1'} \end{aligned}$$

which we rewrite

$$\gamma_1 W_{\Omega_2} = \omega_\lambda [T^\lambda]_{\Omega_2}^{\Omega_1} W_{\Omega_1} = \omega_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_2},$$

where  $W_{\Omega_2} \equiv \mathcal{D}_{\alpha_2} W_{\Omega_1} = \nabla_{\alpha_2} W_{\Omega_1} + K_{\lambda\alpha_2} [T^\lambda]_{\Omega_1}^{\Omega_0} W_{\Omega_0}$ .

Calculating  $\gamma_1 \gamma_1 W_{\Omega_2}$ , we find  $0 = \omega_\alpha \omega_\beta \mathbf{\Gamma}^\alpha \mathbf{\Gamma}^\beta W_{\Omega_2}$ , or  $[\mathbf{\Gamma}^\alpha, \mathbf{\Gamma}^\beta] = 0$ , cf. second equation of (2.18). Also, since

$$W_{\Omega_2} \equiv \mathcal{D}_{\alpha_2} W_{\Omega_1} \equiv \mathcal{D}_{\alpha_2} \mathcal{D}_{\alpha_1} W_{\Omega_0} = (\nabla_{\alpha_2} \nabla_{\alpha_1} + K_{\lambda\alpha_2} [T^\lambda]_{\Omega_1}^{\Omega_0}) W_{\Omega_0},$$

we find that

$$[\mathcal{D}_{\alpha_2}, \mathcal{D}_{\alpha_1}] W_{\Omega_0} = C_{\alpha_2\alpha_1\mu}{}^\nu \Delta_\nu^\mu W_{\Omega_0}, \quad (\text{A.25})$$

in agreement with the second equation of (2.20) and  $\mathbf{\Gamma}^\alpha W_{\Omega_0} = 0$  (equivalent to  $\gamma_1 W_{\Omega_0} = 0$ ).

**(B)** Suppose that we have  $W_{\Omega_k} \equiv \mathcal{D}_{\alpha_k} \dots \mathcal{D}_{\alpha_2} \mathcal{D}_{\alpha_1} W_{\Omega_0}$ ,  $k \geq 2$ . In other words, we know that

$$\begin{aligned} W_{\Omega_k} &= (\nabla_{\alpha_k} + K_{\lambda\alpha_k} \mathbf{\Gamma}^\lambda) W_{\Omega_{k-1}}, \\ \gamma_1 W_{\Omega_k} &= \omega_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_k} \\ &= \omega_\alpha [T^\alpha]_{\Omega_k}^{\Omega_{k-1}} W_{\Omega_{k-1}}, \end{aligned}$$

and

$$\mathbf{\Gamma}^{[\alpha} \mathbf{\Gamma}^{\beta]} W_{\Omega_k} = 0.$$

We want to obtain the next tensor,  $W_{\Omega_{k+1}} \equiv \mathcal{D}_{\alpha_{k+1}} W_{\Omega_k}$ , and its transformation rule.

As before, we first compute the Weyl transformation of  $\nabla_{\alpha_{k+1}} W_{\Omega_k}$ :

$$\begin{aligned} \gamma_1 \nabla_{\alpha_{k+1}} W_{\Omega_k} &= \nabla_{\alpha_{k+1}} \left( \omega_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_k} \right) - \omega_\alpha \mathcal{P}_{\nu\alpha_{k+1}}^{\alpha\mu} \Delta_\mu^\nu W_{\Omega_k} \\ &= (-\gamma_1 K_{\alpha\alpha_{k+1}}) \mathbf{\Gamma}^\alpha W_{\Omega_k} + \omega_\alpha [T^\alpha]_{\Omega_k}^{\Omega_{k-1}} \nabla_{\alpha_{k+1}} W_{\Omega_{k-1}} \\ &\quad - \omega_\alpha \mathcal{P}_{\nu\alpha_{k+1}}^{\alpha\mu} \Delta_\mu^\nu W_{\Omega_k}. \end{aligned}$$

Hence, we get

$$\begin{aligned} \gamma_1 \left( \nabla_{\alpha_{k+1}} W_{\Omega_k} + K_{\alpha\alpha_{k+1}} \mathbf{\Gamma}^\alpha W_{\Omega_k} \right) &= K_{\alpha\alpha_{k+1}} \omega_\beta \mathbf{\Gamma}^\alpha \mathbf{\Gamma}^\beta W_{\Omega_k} \\ &\quad - \omega_\alpha \mathcal{P}_{\nu\alpha_{k+1}}^{\alpha\mu} \Delta_\mu^\nu W_{\Omega_k} + \omega_\alpha [T^\alpha]_{\Omega_k}^{\Omega_{k-1}} \nabla_{\alpha_{k+1}} W_{\Omega_{k-1}}. \end{aligned}$$

Using

$$\nabla_{\alpha_{k+1}} W_{\Omega_{k-1}} = \mathcal{D}_{\alpha_{k+1}} W_{\Omega_{k-1}} - K_{\beta\alpha_{k+1}} \Gamma^\beta W_{\Omega_{k-1}}$$

and posing

$$\mathcal{D}_{\alpha_{k+1}} W_{\Omega_k} = \nabla_{\alpha_{k+1}} W_{\Omega_k} + K_{\alpha_{k+1}\lambda} \Gamma^\lambda W_{\Omega_k},$$

we find

$$\begin{aligned} \gamma_1 \mathcal{D}_{\alpha_{k+1}} W_{\Omega_k} &= K_{\alpha\alpha_{k+1}} \omega_\beta \Gamma^\alpha \Gamma^\beta W_{\Omega_k} - K_{\beta\alpha_{k+1}} \omega_\alpha \Gamma^\alpha \Gamma^\beta W_{\Omega_k} \\ &\quad - \omega_\alpha \mathcal{P}_{\nu\alpha_{k+1}}^{\alpha\mu} \Delta_\mu^\nu W_{\Omega_k} + \omega_\alpha \delta_{\alpha_{k+1}\Omega_{k-1}}^{\Omega'_k} [T^\alpha]_{\Omega_k}^{\Omega_{k-1}} W_{\Omega'_k} \\ &= \omega_\lambda \left( \delta_{\alpha_{k+1}\Omega_{k-1}}^{\Omega'_k} [T^\alpha]_{\Omega_k}^{\Omega_{k-1}} - \delta_{\Omega_k}^{\Omega'_k} \mathcal{P}_{\nu\alpha_{k+1}}^{\lambda\mu} \Delta_\mu^\nu \right) W_{\Omega'_k} \\ &= \omega_\lambda [T^\alpha]_{\alpha_{k+1}\Omega_k}^{\Omega'_k} W_{\Omega'_k}, \end{aligned}$$

where we used  $\Gamma^{[\alpha} \Gamma^{\beta]} W_{\Omega_k} = 0$ .  $\square$

## References

- [1] N. Boulanger and J. Erdmenger, *A classification of local Weyl invariants in  $D = 8$* , Class. Quantum Grav. **21** (2004) 4305–4316 [[hep-th/0405228](#)].
- [2] C. Becchi, A. Rouet and R. Stora, *Renormalization of the abelian Higgs–Kibble model*, Commun. Math. Phys. **42** (1975) 127–162; *Renormalization Of Gauge Theories*, Annals Phys. **98** (1976) 287–321; I. V. Tyutin, *Gauge Invariance In Field Theory And Statistical Physics In Operator Formalism*, LEBEDEV-75-39; J. Zinn-Justin, *Renormalisation of gauge theories*, Lecture notes in Physics n° 37, Springer, Berlin (1975).
- [3] F. Brandt, *Local BRST Cohomology and Covariance*, Comm. Math. Phys. **190** (97) 459–489 [[hep-th/9604025](#)].
- [4] F. Brandt, *Jet coordinates for local BRST cohomology*, Lett. Math. Phys. **55** (2001) 149–159 [[math-ph/0103006](#)].
- [5] N. Boulanger and M. Henneaux, *A derivation of Weyl gravity*, Annalen Phys. **10** (2001) 935–964 [[hep-th/0106065](#)].
- [6] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, 1992.
- [7] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in gauge theories*, Phys. Rept. **338** (2000) 439–569 [[hep-th/0002245](#)].